Supersolid behavior of nonlinear light

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We present a formal demonstration that light can simultaneously exhibit a superfluid behavior and spatial long-range order when propagating in a photonic crystal with self-focussing nonlinearity. In this way, light presents the distinguishing features of matter in a "supersolid" phase. We show that this supersolid phase provides the stability conditions for nonlinear Bloch waves and, at the same time, permits the existence of topological solitons or defects for the envelope of these waves. We use a condensed matter analysis instead of a standard nonlinear optics approach and provide numerical evidence of these theoretical findings.

Matter can undergo a phase transition at ultracold temperatures in which exhibits a superfluid behavior and, at the same time, present all the characteristics of a crystalline solid. This new phase of matter is known as "supersolid" and it has been experimentally proven in helium-4 in recent years [1]. Apparently, light is unrelated to these phases characteristic of condensed matter. However, analogies between condensed matter and optical systems are increasingly appearing in the literature [2]. Even the concept of liquid phase of light has been already suggested [3]. In this letter, we will take this analogy a step further and we will demonstrate a formal equivalence between regular light structures propagating in a nonlinear photonic crystal and ultracold matter in a "supersolid" phase. For the purpose of establishing this equivalence, we will use the formalism of condensed matter and particle physics [4] instead of using a standard nonlinear optics approach.

In Hamiltonian formalism the nonlinear Schrödinger equation for a periodic medium with Kerr nonlinearity can be obtained from the Hamiltonian density as $i\partial\phi/\partial z = \partial\mathcal{H}/\partial\phi^* = \left(-\nabla_t^2 + V(\mathbf{x}) - g|\phi|^2\right)\phi$. In our case ∇_t is the 2D transverse gradient operator and the potential $V(\mathbf{x}) = -\left(n^2(\mathbf{x}) - n_0^2\right)$ represents the periodic modulation of the square of the refractive index with respect to the reference value n_0^2. Transverse and axial coordinates are normalized. The energy of a propagating solution is given by $E = \int d^2x\mathcal{H}(\phi)$. Now, instead of using E we introduce the optical equivalent of the free energy in statistical physics $F = E - \mu P = \int d^2x\mathcal{F}(\phi)$ where μ is the propagation constant of a stationary solution and P is the system power (whose, in absence of losses, is a constant in the propagation). Thus P plays the role of N, the particle number in statistical physics, and μ plays the role of the chemical potential. In terms of the free energy density the equation of motion is $i\partial\bar{\phi}/\partial z = \partial\mathcal{F}/\partial\bar{\phi}^* = \left(-\nabla_t^2 + V(\mathbf{x}) - \mu - g|\bar{\phi}|^2\right)\bar{\phi}$. It is easy to check that the relation between the solutions of both equations is simply $\phi = \bar{\phi}e^{-i\mu z}$. Thus an stationary solution ϕ with propagation constant μ is equivalent to a z-independent solution $\bar{\phi}$ that is an extremum of $\mathcal{F}$ verifying $\partial\mathcal{F}/\partial\bar{\phi}^* = 0$. For the case under consideration, the optical free energy would be given by (we use ϕ instead of $\bar{\phi}$):

$$F = \int d^2x \left[\nabla_t\phi^*\nabla_t\phi + V(\mathbf{x})|\phi|^2 - \mu|\phi|^2 - \frac{g}{2}|\phi|^4\right]. \quad (1)$$

In this context the typical nonlinear optics $P(\mu)$ curve for a stationay solution appears as the conventional equation of state of statistical physics $P = -\partial F/\partial\mu$.

On the other hand, we are interested in the analysis of stationary solutions whose amplitude is invariant under finite translations of value $\mathbf{a}$, where $\mathbf{a}$ is the period of the potential. If $\phi_{\rm sol}$ is a solution that satisfies the equation for stationary states and that simultaneously verifies the condition $|\phi_{\rm sol}(\mathbf{x}+\mathbf{a})| = |\phi_{\rm sol}(\mathbf{x})|$ then the total potential $V_{\rm sol}(\mathbf{x}) \equiv V(\mathbf{x}) - g|\phi_{\rm sol}(\mathbf{x})|^2$ occurring in such equation will be periodic with period given by $\mathbf{a}$, $V_{\rm sol}(\mathbf{x}+\mathbf{a}) = V_{\rm sol}(\mathbf{x})$. Self-consistency implies that $\phi_{\rm sol}$ must be equal to one of the Bloch functions $\phi_{\rm sol}(\mathbf{x}) = f_{\mathbf{Q};\beta_0}(\mathbf{x}) = e^{i\mathbf{Q}\cdot\mathbf{x}}v_{\mathbf{Q};\beta_0}(\mathbf{x})$ forming the spectrum of the nonlinear operator generated by itself and it will be characterized by a propagation constant $\mu = \mu_{\mathbf{Q};\beta_0}$. Using the nonlinear operator generated by $\phi_{\rm sol}$ (including the total potential $V_{\rm sol}(\mathbf{x}) \equiv V(\mathbf{x}) - g|\phi_{\rm sol}(\mathbf{x})|^2$) we construct a *nonlinear* Wannier function basis by means of standard techniques. We assume that all functions are defined within a periodic region Ω, called the basic domain, of spatial dimensions $Na \times Na$ so that we represent any arbitrary field amplitude at a given axial position z as $\phi(\mathbf{x},z) = \sum_{\hat{\imath},\alpha} c_{\hat{\imath},\alpha}(z)W^{\mu}_{\alpha}(\mathbf{x}-\mathbf{x}_{\hat{\imath}})$.The nonlinear Bloch solution $\phi_{\rm sol}(\mathbf{x})$ is represented by the site-independent and z-independent coefficients $c^{\rm sol}_{\hat{\imath}} = \eta_\mu e^{i\mathbf{Q}\cdot\mathbf{x}_{\hat{\imath}}}$ where $\eta_\mu \equiv \sqrt{P}/N$. Note that $\mathbf{Q}$ is discretized because ϕ is defined in the periodic domain Ω so that $\mathbf{Q_m} = 2\pi\mathbf{m}/(Na)$ where $\mathbf{m} \in \mathbb{Z}^2$ and fulfills $-N/2 \leq m_x, m_y \leq N/2$. We introduce now the Wannier expansion into the expression for the optical free energy and integrate out the tranverse

coordinates $\mathbf{x}$ to obtain

$$F = -t(\mu)\sum_{\hat{i}}\sum_{\nu=1}^{2} c^*_{\hat{i}}\left(c_{\hat{i}+\hat{n}_\nu} + c_{\hat{i}-\hat{n}_\nu}\right) + \\ +l(\mu)\sum_{\hat{i}}|c_{\hat{i}}|^2 - \frac{U(\mu)}{2}\sum_{\hat{i}}|c_{\hat{i}}|^4 + (\text{h.o.t.})\,, \quad (2)$$

where $\hat{n}_1$ and $\hat{n}_2$ are the "horizontal" and "vertical" lattice vectors, $t(\mu) \equiv -L_{\hat{0},\hat{0}+\hat{n}_1} = -L_{\hat{0},\hat{0}+\hat{n}_2}$ is the effective nearest neighbors coupling and $l(\mu) \equiv L_{\hat{0}\hat{0}}$ and $U(\mu) \equiv 2T_{\hat{0}\hat{0}\hat{0}\hat{0}}$ are the two on-site couplings. The second and fourth order couplings $L_{\hat{i}\hat{j}}(\mu)$ and $T_{\hat{i}\hat{j}\hat{k}\hat{l}}(\mu)$ are obtained as overlapping integrals of the nonlinear Wannier basis associated to the stationary solution and, for this reason, they depend on $(\mu, \mathbf{Q}, \beta_0)$ (for simplicity, we disregard interband interaction terms and elliminate band indices): $L_{\hat{i}\hat{j}}(\mu) \equiv \int_\Omega d^2x W^{\mu *}_{\hat{i}}\left(-\nabla^2_t + V(\mathbf{x}) - \mu\right) W^{\mu}_{\hat{j}}$ and $T_{\hat{i}\hat{j}\hat{k}\hat{l}}(\mu) \equiv \frac{1}{2}\int_\Omega d^2x g W^{\mu *}_{\hat{i}} W^{\mu *}_{\hat{j}} W^{\mu}_{\hat{k}} W^{\mu}_{\hat{l}}$. The expression (h.o.t.) in (2) stands for higher-order terms involving interactions at longer distances.

The main difference between (2) and previous approaches is that the nonlinear Wannier basis provide different coefficients than those obtained using linear Wannier functions or localized single potential solutions as in the tight binding aproximation. Since our aim is to analize the stability behavior of the nonlinear Bloch solution characterized by μ the election of the nonlinear Wannier basis associated to it is a natural choice. With this purpose in mind let us write the z-dependent Wannier coefficients as $c_i(z) = e^{i\mathbf{Q}\cdot\mathbf{x}_{\hat{i}}}\Phi_i(z) = e^{i\mathbf{Q}\cdot\mathbf{x}_{\hat{i}}}\left(\eta_\mu + \Delta\Phi_i(z)\right)$ to formalize the fact that we want to analyze the dynamics associated to the stationary solution described by $c^{\rm sol}_{\hat{i}} = \eta_\mu e^{i\mathbf{Q}\cdot\mathbf{x}_{\hat{i}}}$. In this way the dynamic information is encoded in the envelope coefficients $\Phi_i(z)$.

In this Letter we are interested in the regime where perturbations have a correlation length larger than the potential period —$\zeta \gg a$. Physical phenomena in this regime present a collective spatial character that will reflected in the properties of the discrete envelope function $\Phi_i(z)$. Spatial collective effects will be represented by values of Φ_i that will fluctuate smoothly in space. It is then natural to take the continuous limit of the discrete optical free energy by considering the limit $a/\zeta \to 0$ and by introducing the continuous spatial envelope function $\Phi(\mathbf{x}_t, z)$ defined as $\Phi(\mathbf{x}_{\hat{i}}, z) = \Phi_i(z)$. We substitute $c_i = e^{i\mathbf{Q}\cdot\mathbf{x}_{\hat{i}}}\Phi_i$ into (2) and then take the continuous limit using standard techniques. In order to simplify the result, we consider only "diagonal" nonlinear Bloch waves with $Q_1 = Q_2 = 2\pi m/(Na)$, $m \in \mathbb{Z}$. The phase difference between two neighboring sites of the nonlinear Bloch wave is then $\Delta\phi_m = 2\pi m/N$. In order to relate the optical effects driven by F to condensed matter and particle physics phenomena we need to re-define the optical free energy in some cases. For this reason, we introduce the sign-changed optical free energy defined as

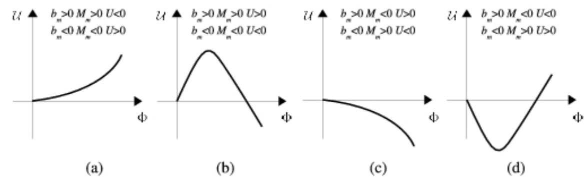

Figure 1: Effective potential for different configurations

$$\bar{F} \equiv \text{sign}(b_m)F$$

$$\bar{F} = \int_\Omega d^2x \left[|b_m(\mu)|\nabla\Phi^*\nabla\Phi + i\,\text{sign}(b_m)(\mathbf{v}_m(\mu)\cdot\nabla\Phi)\Phi^* \right. \\ \left. +\mathcal{U}(|\Phi|^2) + (\text{h.o.t.})\right]\,, \quad (3)$$

where $b_m(\mu) \equiv \cos(\Delta\phi_m)t(\mu)a^2$, $\mathbf{v}_m(\mu) = v_m(1,1)$ with $v_m(\mu) \equiv -2\sin(\Delta\phi_m)t(\mu)a$ and $M_m(\mu) \equiv l(\mu) - 4t(\mu)\cos(\Delta\phi_m)$. In this case (h.o.t.) includes terms with higher-order derivatives and $\mathcal{U} = \text{sign}(b_m)\left(M_m(\mu)|\Phi|^2 - \frac{U(\mu)}{2}|\Phi|^4\right)$. In principle, the use of F or $\bar{F}$ is irrelevant as far as dynamics is concerned since dynamics remains the same under the change $F \to -F$. The advantage in the use of $\bar{F}$ is that it contains *in all cases* a positive definite kinetic energy term, as required in quantum field theory to define a proper vacuum state [5]. By doing this, equivalences are straightforward to achieve even with negative values of b_m. By means of $\bar{F}$ we can proceed to determine the nature of the ground state of the system by analyzing the behavior of the so-called effective potential $U(\Phi)$ "à la Landau". Landau theory is a mean-field approach used to characterize phase transitions in condensed matter and particle physics [4]. The qualitative form of the effective potential is represented in Fig. (1) for different signs of coefficients. One can easily recognize in this figure that there are only two configurations for which a spatially uniform envelope solution $\Phi(\mathbf{x}) = \Phi_0$ is allowed (cases (b) and (d)). They correspond to extrema of the optical free energy density $\partial\mathcal{F}/\partial\Phi^* = \partial\mathcal{U}/\partial\Phi^* = 0$ given by the condition $|\Phi_0| = M_m(\mu)/U(\mu)$. The ground state of the system is the state that minimizes the free energy, thus, only the two cases in (d) can provide an spatially homogeneous ground state. The former analysis is identical to that performed in condensed matter physics to establish the nature of phase transitions in the mean-field regime, here μ playing the role of temperature. In this way, the signs of the $b_m(\mu)$, $M_m(\mu)$ and $U(\mu)$ coefficients determine the nature of the ground state and, therefore, in which*"phase"* light is. It is important to strees that the μ dependence in these "Landau coefficients" is the result of using nonlinear Wannier functions. The ground state in Fig.1(d) is degenerate since all solutions of the type $\Phi_\phi = |\Phi_0|e^{i\phi}$ are minima of the effective potential and thus they all have the same free energy. The

optical free energy is invariant under a $U(1)$ phase transformation $\Phi \to e^{i\alpha}\Phi$. The ground state Φ_ϕ, however, is not since this phase transformation maps it into a different degenerate solution $\Phi_{\phi+\alpha}$ with the same free energy. This mechanism is well-known in condensed matter and particle physics and it is known as *spontaneous symmetry breaking* (SSB). Superfluidity, superconductivity, the Higgs mechanism or the chiral phase transition in quantum chromodynamics are physical phenomena related to the same mechanism. The SSB mechanism have distinctive properties: (i) appearence of a non-zero order parameter in the broken phase, (ii) existence of topological solitons or defects, (iii) presence of masless or long-range excitations (Goldstone bosons). In this language, our optical system in the Fig.1(d) configuration is in a *broken phase*, a phase in which $U(1)$ symmetry has been spontaneously broken. We expect then to find the optical counterparts of the aforementioned properties. In this Letter we will pay attention to properties (i) and (ii), leaving the analysis of (iii) for a further publication.

We can now establish a link between the stability of nonlinear Bloch waves and the SSB mechanism. A nonlinear Bloch wave is characterized by an homogeneous envelope. According to our previous analysis, only in the configurations in Fig.1(d) this envelope function corresponds to the ground state of the optical free energy. On the other hand, in this configuration the system is in the $U(1)$ broken phase. Since the ground state is a stable solution, i.e., fluctuations around it cannot transform this solution into a different one by any dynamical mechanism, we infer that the stability condition of a nonlinear Bloch wave is achieved in the broken phase, i.e., when ($b_m > 0$, $M_m < 0$, $U < 0$) or ($b_m < 0$, $M_m > 0$, $U > 0$). The properties of a stable nonlinear Bloch wave are then identical to that of a superfluid as far as its envelope is concerned. However, the solution simultaneously presents spatial long-range order. Using quantum field theory notation, if $|0\rangle$ stands for the ground state given by the nonlinear Bloch wave in the mean-field regime at a given μ: (i) $T_{\mathbf{a}}\,|0\rangle = e^{i\mathbf{Q}\cdot\mathbf{a}}\,|0\rangle$ where $T_{\mathbf{a}}$ is the lattice translation operator and (ii) $\langle 0|\,\hat{\Phi}\,|0\rangle = |\Phi_0|e^{i\phi} \neq 0$. Physically speaking, the "soliton lattice" filling the periodic medium completely and described by the nonlinear Bloch wave presents perfect spatial long-range order as in a cristallyne solid and, at the same time, it possesses a non-vanishing order parameter indicating it is in a superfluid phase. The dynamics of low energy fluctuations around such a solution is determined by Eq.(3) and has to show identical features than a superfluid. In this way, we demonstrate that, under the specified conditions, light fulfills the definition of a supersolid, that is, it is a spatially ordered system (like in a solid or crystal) with superfluid properties.

We have performed a number of numerical experiments to check both qualitative and quantitative the validity of our condensed matter approach. We have modelled a nonlinear photonic crystal with $\mathcal{C}_4$ symmetry formed by a nonlinear material with refractive index n_1 with embedded circular inclusions of a linear material with index $n_2 < n_1$, the index contrast being given by the potential $V_0 = -\left(n_2^2 - n_1^2\right) > 0$. This type of nonlinear photonic crystal can be achieved in standard or in chalcogenide photonic crystal fibers [6] or in laser-writing waveguides [7]. The nonlinearity is self-focussing and of the Kerr type $V_{\rm NL} = -g|\phi|^2$ ($g > 0$). First of all, we have evaluated the dependence of the "Landau coefficients" (b_m, M_m, U) on μ . We have proceeded as follows: (i) we numerically evaluate a nonlinear Bloch solution $\phi_{\rm sol}$ at a given μ, (ii) we determine the nonlinear operator associated to $\phi_{\rm sol}$ (i.e., including the nonlinear potential $V_{\rm NL} = -g|\phi_{\rm sol}|^2$) which numerically will be a matrix, (iii) we find the spectrum of this matrix formed by Bloch eigemodes since the

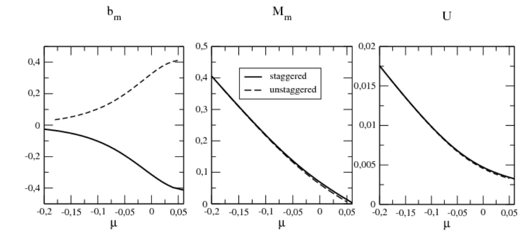

Figure 2: Landau coefficients (b_m, M_m, U) in terms of the propagation constant μ for unstaggered (dashed line) and staggered (solid line) nonlinear Bloch solutions.

(a) (b)

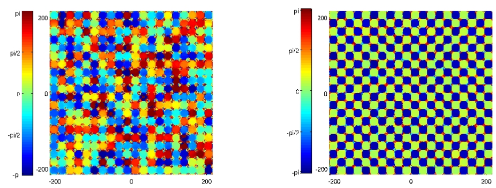

Figure 3: Stability/unstability patterns of phase under propagation:(a) for a perturbed unstaggered solution phase disorders quickly, (b) for a staggered solution spatial order remains.

(a) (b) (c)

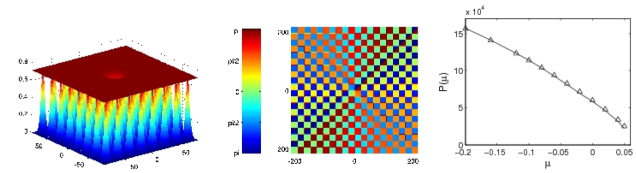

Figure 4: Supersolid light topological defect for $\mu = -0.1$, $V_0 = 2$, and charge $+1$: (a) simultaneous representation of the amplitudes of the envelope Φ and full ϕ functions for this solution, (b) representation of its phase and (c) $P(\mu)$ diagram calculated using the envelope equation (solid line) and the full equation (triangles).

total potential, including the linear and nonlinear part, is periodic, (iv) we construct the *nonlinear* Wannier basis out of these Bloch modes using the standard technique [8] and (v) we evaluate the "Landau coefficients" using their definitions as overlapping integrals of Wannier functions. A given nonlinear Bloch wave characterized by μ then univocally provides a specific value for (b_m, M_m, U). In Fig. 2 we represent the functional form of these coefficients in terms of μ for two solutions with different pseudo-momentum $\mathbf{Q}$: a solution with $\mathbf{Q} = 0$ with a phase difference between neighboring sites $\Delta\phi_m = 0$ (unstaggered) and a solution with $\mathbf{Q} = (\pi/a, \pi/a)$ with $\Delta\phi_m = \pi$ (staggered). By analyzing the signs of their Landau coefficients we immediately recognize these two configurations correspond to the cases (b) (unstaggered) and (d) (staggered) in Fig.1. The main difference between these two configurations arise from the different sign of b_m, which has its origin in their different phase structure since $b_m \sim t(\mu)\cos(\Delta\phi_m)$. In the case analyzed here $t(\mu) > 0$ for all μ in both configurations, so that the different form of the effective potential in both cases is a pure effect of the underlying phase of the nonlinear Bloch wave. According to our condensed matter analysis, the unstaggered solution can exist as an homogeneous solution but it corresponds to a metastable state that eventually will decay into a different state. It cannot be stable. On the contrary, the staggered solution is the ground state of a light supersolid and, consequently, it is necessary stable. We have performed a numerical stability analysis of these two configurations that confirm these predictions. A small initial perturbation of the unstaggered solution gives rise, after a short propagation, to the breaking of spatial long-range order crearly reflected in the progressive disordering of the phase of the solution —see Fig.3(a). The staggered solution is, however, immune to perturbations and preserve spatial long-range order in phase and amplitude —see Fig.3(b). Similar examples can be found in the literature that can be explained by the same mechanism, e.g., stability of large truncated nonlinear Bloch waves [9]. As mentioned before, one of the most paradigmatic properties of a superfluid is the existence of topological solitons or defects. They are holes in the superfluid around which the superfluid flows with a quantized circulation. Therefore, it is expected that the envelope function Φ supports the existence of the optical counterparts of these objects in the broken phase. We have proven indeed the existence of these type of solutions by numerically solving the same nonlinear photonic crystal structure using: (i) the *effective* equation for Φ associated to $\bar{F}$ —Eq.(3)— using the values of (b_m, M_m, U) given in Fig.(2) and (ii) the full equation for the original field ϕ including the periodic potential $V(\mathbf{x})$. It is remarkable that both approaches are in excellent agreement with each other both qualitatively and quantitatively. As correctly predicted by theory, the topological vortex only exists *on top of* the staggered solution —see Fig.4(b)— which corresponds to the ground state in the broken superfluid phase $|\Phi_0| \neq 0$. From an optical point of view, one could consider this object as a type of dark soliton, however, the striking property of this dark soliton is that exists in a *self-focusing* medium when they are usually associated to defocusing media. In Fig.4(a) we can appreciate the excellent fit of the envelope function to the solution of the full equation. This excellent quantitative agreement is even more clearly appreciated in the calculation of the $P(\mu)$ curve for a family of topological vortices using both approaches shown in Fig.4(c). In summary, Figs.4(a) and (b) show the simultaneous presence of superfluid behavior (nontrivial amplitude and phase of the topological defect) and spatial long-range order (perfect staggered order of the background in all the domain). Besides, numerical stability analysis indicates that these solutions are stable under propagation during long distances. These features represent a clear numerical evidence of the supersolid behavior of light in nonlinear photonic crystals predicted by theory. In this sense, it has been shown that light can be treated analogously as matter not only in their qualitative aspects but by using a condensed matter formalism. We call this approach to nonlinear optics *photonic condensed matter.*

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